

Interplay between interaction and chiral anomaly in the holographic approach

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Strongly coupled conformal field theory appears to describe universal scaling around quantum criticality, where critical exponents reflect the nature of emergent excitations. In particular, novel symmetries can emerge from strong interactions, expected to be responsible for quantum number fractionalization. The underlying mechanism has been proposed that an emergent enhanced symmetry allows a topological term associated with anomaly, which assigns a fermion's quantum number to a topological excitation, referred as the Goldstone-Wilczek current. Although this mechanism has been verified in one dimensional interacting electrons, where either spinons or holons are identified with topological solitons, its generalization to higher dimensions is beyond the field theoretic framework in the respect that interplay between interaction and anomaly cannot be taken into account sincerely within the field theory technique above one dimension. In this study we examine the interplay between correlations and topological terms based on the holographic approach, allowing us to incorporate such nonperturbative quantum effects via solving classical equations of motion but on a curved space. We solve the Einstein-Maxwell-Chern-Simons theory on the Reissner-Nordström-AdS₅ in the extremal limit, and uncover novel critical exponents to appear in the current-current correlation functions, where both the emergent locality and the Chern-Simons term play an important role in such critical exponents. We speculate that the corresponding conformal field theory may result from interacting $U(1)$ currents with chiral anomaly at finite density, expected to be applicable to topological insulators with strong interactions, where dyon-type excitations appear to carry nontrivial fermion's quantum numbers.

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I. INTRODUCTION

Universal scaling is an essential aspect in critical phenomena, where critical exponents imply the information of emergent excitations [1, 2]. Elementary excitations become strongly coupled in the vicinity of quantum criticality, described by conformal field theories (CFTs), which do not allow well defined excitations [3, 4]. One can consider two possible scenarios for disappearance of quasi-particles. One immediately suggests that elementary excitations decay into bunch of soft modes, given by composite particles in terms of original excitations. This corresponds to usual situations in critical phenomena [5]. On the other hand, elementary excitations may become fractionalized into more “basic” ingredients, which also leads the quasiparticle weight to vanish. When quantum number fractionalization occurs, critical exponents are enhanced, originating from the fact that elementary excitations are given by composite particles in terms of such fractionalized excitations [6].

The second scenario is actually well known in one dimensional correlated systems, where such fractionalized excitations are referred as spinons and holons [7]. The underlying mechanism has been proposed that novel symmetries can emerge from strong interactions, expected to be responsible for novel quantum numbers. The emergent enhanced symmetry allows a topological term associated with anomaly, which assigns a fermion's quantum number to a topological excitation, referred as the Goldstone-Wilczek current [8]. However, its generaliza-

tion to higher dimensions [6] is beyond the field theoretic framework because interplay between interaction and anomaly cannot be taken into account sincerely within the field theory technique above one dimension. Only several simulation results have been reported in extended Heisenberg models [9] and some statistical field theories [10].

In string theory, it has been clarified that strongly coupled CFTs in d -dimensions can be mapped into classical gravity theories on Anti-de Sitter space in $(d+1)$ -dimensions (AdS _{$d+1$}) [11, 12]. This duality may be regarded as generalization of the well known holography of the bulk-edge correspondence in the quantum Hall effect, where this framework has been developed in the context of string theory, referred as the AdS/CFT correspondence. See Ref. [13] for a review. Immediately, it has been applied to various problems beyond techniques of field theories: nonperturbative phenomena in quantum chromodynamics (AdS/QCD or holographic QCD) [14], non-Fermi liquid transport near quantum criticality [15–17] and superconductors [18–20] in condensed matter physics (AdS/CMP), and etc.

The motivation for the usage of the AdS/CFT machinery lies in solving two difficult problems beyond the field theoretical framework. Although vector models are expected to be under control generally, interacting fermions at finite density, named as the Fermi surface problem, turns out to be out of control, which displays essentially the same aspect as the matrix model, where the infinite limit of the flavor degeneracy is impossible to be performed within the present technology because all pla-

nar diagrams are exactly at the same order and they all should be summed [21–25]. In addition to the treatment of strong correlations, we do not know how to take into account the topological term nonperturbatively above one dimension, as discussed before. The AdS/CFT duality tells us that we can incorporate both strong correlations and topological effects just via solving classical equations of motion, but in one dimension higher than that of the field theory and on the curved space.

In this study we examine the interplay between correlations and topological terms based on the holographic approach. We solve the Einstein-Maxwell-Chern-Simons (EM-CS) theory on the Reissner-Nordström-AdS₅ (RN-AdS₅) in the extremal limit, and uncover novel critical exponents to appear in the current-current correlation functions, where both the emergent locality and the Chern-Simons term play an important role in such critical exponents. We speculate that the corresponding conformal field theory may result from strongly interacting $U(1)$ charge currents at finite density in one time and three space dimensions. In particular, such a field theory is expected to contain the topological θ term, associated with anomalous “chiral” currents. We suggest that strong correlations in the presence of θ vacua may result in novel emergent excitations, reflected in their critical exponents and distinguished from “boring” excitations in the absence of the θ term. We discuss that such a conformal field theory may appear at quantum criticality in topological insulators with fractional magnetoelectric effect [26]. However, we cannot interpret the emergence of locality in the field theoretic point of view, which occurs in the case when the dynamical critical exponent becomes infinite. Although it was demonstrated that the dynamical critical exponent can be changed due to higher-loop quantum corrections [23], the infinite dynamical exponent does not seem to be reachable within the present technology except for the disorder-driven Anderson localization [27][46].

II. EINSTEIN-MAXWELL-CHERN-SIMONS

We start by giving 5D gravity side description which may be dual to 4D field theory. In addition to the usual Einstein gravity, we consider the 5D Maxwell field which could couple to the $U(1)$ current on the boundary. We can also introduce the CS term in the 5D spacetime. The action which we would like to work with is the EM-CS in 5D with the gravitational constant G_5 , the negative cosmological constant $\Lambda (= -6/l^2)$, the gauge coupling e^2 and the Chern-Simons coupling κ :

$$S = S_{\text{EH}} + S_{\text{GH}} + S_{\text{Maxwell}} + S_{\text{CS}}, \quad (1)$$

where

$$S_{\text{EH}} = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} (R - 2\Lambda), \quad (2a)$$

$$S_{\text{GH}} = \frac{1}{8\pi G_5} \int d^4x \sqrt{-g^{(4)}} K, \quad (2b)$$

$$S_{\text{Maxwell}} = -\frac{1}{4e^2} \int d^5x \sqrt{-g} \mathcal{F}_{mn} \mathcal{F}^{mn}, \quad (2c)$$

$$S_{\text{CS}} = \frac{\kappa}{3} \int d^5x \varepsilon^{lmnpq} \mathcal{A}_l \mathcal{F}_{mn} \mathcal{F}_{pq}. \quad (2d)$$

In order to compensate well-defined variational principle for the Einstein-Hilbert action S_{EH} with boundary, we have introduced the Gibbons-Hawking term S_{GH} in which $g_{\mu\nu}^{(4)}$ and K are the induced metric and the extrinsic curvature on the 4D boundary, respectively. Throughout of this paper, we follow the notation given in the previous work [28].

A. RN-AdS₅ background

Equations of motion read

$$R_{mn} - \frac{1}{2} g_{mn} R - \frac{6}{l^2} g_{mn} = 8\pi G_5 T_{mn}, \quad (3a)$$

$$-\frac{1}{e^2} \nabla_n \mathcal{F}^{mn} + \frac{\kappa}{\sqrt{-g}} \varepsilon^{mlnpq} \mathcal{F}_{ln} \mathcal{F}_{pq} = 0, \quad (3b)$$

where T_{mn} is the energy-momentum tensor,

$$T_{mn} = \frac{1}{e^2} \left(\mathcal{F}_{mk} \mathcal{F}_{nl} g^{kl} - \frac{1}{4} g_{mn} \mathcal{F}_{kl} \mathcal{F}^{kl} \right). \quad (4)$$

RN-AdS₅ background with AdS radius l is a solution of the equations of motion (3a) and (3b) even in the presence of the CS term:

$$(ds)^2 = \frac{r^2}{l^2} \left(-f(r) (dt)^2 + (d\vec{x})^2 \right) + \frac{l^2}{r^2 f(r)} (dr)^2, \quad (5)$$

$$\mathcal{A}_t(r) = -\frac{Q}{r^2} + \mu,$$

with

$$\begin{aligned} f(r) &= 1 - \frac{ml^2}{r^4} + \frac{q^2 l^2}{r^6} \\ &= \frac{1}{r^6} (r^2 - r_0^2)(r^2 - r_+^2)(r^2 - r_-^2), \end{aligned}$$

and

$$q = 4\sqrt{\frac{\pi G_5}{3e^2}} Q.$$

The parameters m and q correspond to the mass and charge of the AdS space, respectively. The asymptotic value of gauge field $\mathcal{A}_t(r \rightarrow \infty) = \mu$ may be interpreted

as the chemical potential in the dual field theory. The explicit forms of $r_0 (= -r_+^2 - r_-^2)$ and r_\pm are given by

$$\begin{aligned} r_0^2 &= \left(\frac{m}{3q^2} \left(1 + 2 \cos \left(\frac{\theta}{3} + \frac{2}{3}\pi \right) \right) \right)^{-1}, \\ r_+^2 &= \left(\frac{m}{3q^2} \left(1 + 2 \cos \left(\frac{\theta}{3} + \frac{4}{3}\pi \right) \right) \right)^{-1}, \\ r_-^2 &= \left(\frac{m}{3q^2} \left(1 + 2 \cos \left(\frac{\theta}{3} \right) \right) \right)^{-1}, \end{aligned}$$

with

$$\theta = \arctan \left(\frac{3\sqrt{3}q^2 \sqrt{4m^3 l^2 - 27q^4}}{2m^3 l^2 - 27q^4} \right),$$

where r_+ and r_- represent locations of the outer and inner horizons, respectively.

The Hawking temperature of RN-AdS₅ background which may correspond to the temperature of the dual field theory is given as

$$T = \frac{r_+^2 f'(r_+)}{4\pi l^2} = \frac{r_+}{\pi l^2} \left(1 - \frac{1}{2} \frac{q^2 l^2}{r_+^6} \right) \equiv \frac{1}{2\pi b} \left(1 - \frac{a}{2} \right), \quad (6)$$

where a and b are defined as

$$a \equiv \frac{q^2 l^2}{r_+^6} \quad b \equiv \frac{l^2}{2r_+}.$$

The value of a can be taken in $0 \leq a \leq 2$. The entropy density s , the energy density ϵ , the chemical potential and the physical charge density ρ in the dual field theory are given by

$$s = \frac{r_+^3}{4G_5 l^3}, \quad \epsilon = \frac{3m}{16G_5 l^3}, \quad \mu = \frac{Q}{r_+^2}, \quad \rho = \frac{2Q}{e^2 l^3},$$

respectively. This kind of charged AdS background has been used in various contexts in the AdS/CFT correspondence [29].

It might be convenient to introduce new dimensionless coordinate $u \equiv r_+^2/r^2$ which is normalized by the outer horizon. In this coordinate, the horizon and the boundary are located at $u = 1$ and $u = 0$, respectively. The background (5) can be rewritten as

$$\begin{aligned} (ds)^2 &= \frac{l^2}{4b^2 u} \left(-f(u)(dt)^2 + (d\vec{x})^2 \right) \\ &\quad + \frac{l^2}{4u^2 f(u)} (du)^2, \\ \mathcal{A}_t(u) &= \mu(1-u), \end{aligned} \quad (7)$$

with

$$f(u) = (1-u)(1+u-au^2).$$

B. Zero temperature and AdS₂

We now consider the extremal limit i.e.

$$a = 2.$$

Through the definition (6), we could access to the zero temperature system for the dual field theory. Hereafter we focus on this extremal zero temperature case. In this system, the only physical parameter in the bulk solution is the chemical potential μ .

Near the horizon $u = 1$ in the extremal RN-AdS₅, the AdS₂ structure emerges [30]. Near the horizon, we introduce dimensionless coordinates (τ, ζ) as

$$1-u = \frac{\epsilon}{\zeta}, \quad t = \frac{\alpha \tau}{\mu \epsilon}, \quad (8)$$

where we have introduced the dimensionless constant $\alpha = el/(4\sqrt{6\pi G_5})$ to make some expression simpler. Taking the scaling limit $\epsilon \rightarrow 0$ with finite (τ, ζ) , the background (7) becomes

$$\begin{aligned} (ds)^2 &= \frac{l^2}{12\zeta^2} \left(-(d\tau)^2 + (d\zeta)^2 \right) + \frac{l^2}{4b^2} (d\vec{x})^2, \\ A_\tau(\zeta) &= \frac{\alpha}{\zeta}, \end{aligned} \quad (9)$$

which gives AdS₂ × R³. We refer $\zeta \rightarrow \infty$ as the Poincaré horizon, while $\zeta \rightarrow 0$ as the AdS₂ boundary where the IR CFT could be defined.

III. PERTURBATIONS OF RN-ADS₅ BACKGROUND

Now we consider perturbations on the extremal RN-AdS₅ background,

$$g_{mn} = g_{mn}^{(0)} + h_{mn} \quad \text{and} \quad \mathcal{A}_m = A_m^{(0)} + A_m,$$

where $(g_{mn}^{(0)}(u), A_m^{(0)}(u))$ and $(h_{mn}(u, x^\mu), A_m(u, x^\mu))$ denote the extremal RN-AdS₅ background (7) and the perturbations, respectively. We choose the following gauge conditions,

$$h_{um} = 0 \quad \text{and} \quad A_u = 0,$$

and use Fourier expansion in which the momentum lies on the z -direction,

$$\begin{aligned} h_{\mu\nu}(t, z, u) &= \int \frac{d^2 k}{(2\pi)^2} e^{-i\omega t + ikz} h_{\mu\nu}(\omega, k, u), \\ A_\mu(t, z, u) &= \int \frac{d^2 k}{(2\pi)^2} e^{-i\omega t + ikz} A_\mu(\omega, k, u), \end{aligned}$$

where μ and ν run through 4D spacetime except for the radial direction. At the boundary, these fluctuations $(h_{\mu\nu}(u, x^\mu), A_\mu(u, x^\mu))$ may couple to the operators

$(T^{\mu\nu}(x^\mu), J^\mu(x^\mu))$ in the boundary field theory, where $T^{\mu\nu}(x^\mu)$ and $J^\mu(x^\mu)$ are the energy momentum tensor and $U(1)$ charge current, respectively. By using Gubser-Klebanov-Polyakov-Witten (GKP-W) relation [12], one can calculate correlation functions through evaluating the bulk on-shell action. As we saw, the background solution does not depend on the existence of the CS term. On the other hand, fluctuations are affected by the CS term. The effect of the CS term to the dual field theory is our main topic.

The perturbations can be categorized to the three types i.e. scalar, vector and tensor types by using the spin under the $O(2)$ rotation in the (x, y) -plane [31]. It is easy to show that the CS term contributes only to the vector type perturbation whose non-zero values are listed below:

$$h_{xt}, \quad h_{yt}, \quad h_{xz}, \quad h_{yz}, \quad \text{and} \quad A_x, \quad A_y.$$

It might be convenient to work with the variables $h_t^x(u) = g^{(0)xx}h_{xt}(u)$, $h_z^x(u) = g^{(0)xx}h_{xz}(u)$, $h_t^y(u) = g^{(0)yy}h_{yt}(u)$ and $h_z^y(u) = g^{(0)yy}h_{yz}(u)$.

A. Decoupling of the equation of motion

The equations of motion (3a) and (3b) for the perturbation fields are given by

$$0 = h_t^{x(y)''} - \frac{1}{u}h_t^{x(y)'} - \frac{9}{uf(u)}\left(\omega\mathbf{k}h_z^{x(y)} + \mathbf{k}^2h_t^{x(y)}\right) - 6uB'_{x(y)}, \quad (10a)$$

$$0 = \mathbf{k}f(u)h_z^{x(y)'} + \omega h_t^{x(y)'} - 6\omega uB_{x(y)}, \quad (10b)$$

$$0 = h_z^{x(y)''} + \frac{(u^{-1}f(u))'}{u^{-1}f(u)}h_z^{x(y)'} + \frac{9}{uf^2(u)}\left(\omega^2h_z^{x(y)} + \omega\mathbf{k}h_t^{x(y)}\right), \quad (10c)$$

and

$$0 = B''_{x(y)} + \frac{f'(u)}{f(u)}B'_{x(y)} + \frac{9}{uf^2(u)}\left(\omega^2 - \mathbf{k}^2f(u)\right)B_{x(y)} - \frac{1}{f(u)}h_t^{x(y)'} - (+)\tilde{\kappa}\frac{i\mathbf{k}}{f(u)}B_{y(x)}, \quad (11)$$

with

$$f(u) = (1-u)^2(1+2u), \quad \tilde{\kappa} \equiv \frac{(12e)^2\alpha}{l}\kappa, \quad B_{x(y)} \equiv \frac{A_{x(y)}}{\mu} = \frac{b}{3\alpha}A_{x(y)},$$

where the prime implies the derivative with respect to u . We have normalized the frequency and momentum by

the chemical potential $\omega \equiv \alpha\omega/\mu$ and $\mathbf{k} \equiv \alpha\mathbf{k}/\mu$, respectively. In the presence of the CS coupling $\tilde{\kappa}$, the x - and y -components of gauge fields are coupled. As we explain below, there are four independent variables. Eq. (10c) can be derived from (10a) and (10b). $h_z^{x(y)'}(u)$ could be expressed in terms of $h_t^{x(y)'}(u)$ and $B_{x(y)}(u)$ through (10b). From (10a) and (10b) we can obtain a second order differential equation for $h_t^{x(y)'}(u)$ with $B_{x(y)}(u)$. Together with (11), we treat $h_t^{x(y)'}(u)$ and $B_{x(y)}(u)$ as the four independent variables.

In order to solve the coupled equations of motion, it might be convenient to introduce master variables [32]. By using master variables given in [28], which correspond to the helicity bases on the (x, y) -plane [33], the equations of motion are decoupled and organized as four ordinary differential equations. In the matrix form, the master equations are given by

$$0 = \tilde{\Theta}'' + \frac{(u^2f(u))'}{u^2f(u)}\tilde{\Theta}' + \tilde{\Omega}(u)\tilde{\Theta}, \quad (12)$$

with potential

$$\tilde{\Omega}(u) = \frac{9}{uf^2(u)}\omega^2 + \frac{1}{2uf(u)}\left((\mathcal{D}_{\tilde{\kappa}}(\mathbf{k}) - 6)u - 18\mathbf{k}^2\right). \quad (13)$$

The diagonal matrix $\mathcal{D}_{\tilde{\kappa}}(\mathbf{k})$ is given as

$$\mathcal{D}_{\tilde{\kappa}}(\mathbf{k}) = \text{diag}\left(-D_- + \tilde{\kappa}\mathbf{k}, D_- + \tilde{\kappa}\mathbf{k}, -D_+ - \tilde{\kappa}\mathbf{k}, D_+ - \tilde{\kappa}\mathbf{k}\right), \quad (14)$$

with constants D_\pm defined through,

$$\begin{aligned} C_\pm &\equiv 3 \pm 3\sqrt{1+6\mathbf{k}^2}, \\ C_0 &\equiv C_+ - C_-, \\ D_\pm &\equiv \sqrt{(C_0 \pm \tilde{\kappa}\mathbf{k})^2 \pm 4\tilde{\kappa}\mathbf{k}C_-} \\ &= \sqrt{\tilde{\kappa}^2\mathbf{k}^2 + 216\mathbf{k}^2 \pm 12\tilde{\kappa}\mathbf{k} + 36}. \end{aligned}$$

Here the master variables $\tilde{\Theta}_a(u)$ are related with the original variables via,

$$\tilde{\Theta}_a = (\Lambda^{-1}\Theta)_a, \quad (15)$$

where $\Theta_a \equiv (\Theta_{x+}, \Theta_{x-}, \Theta_{y+}, \Theta_{y-})^T \equiv (\Theta_1, \Theta_2, \Theta_3, \Theta_4)^T$ are given as

$$\Theta_{x(y)\pm} = \frac{1}{u}h_t^{x(y)'} - \left(6 - \frac{C_\pm}{u}\right)B_{x(y)}. \quad (16)$$

The constant matrix Λ^{-1} is expressed as

$$\Lambda^{-1} = \begin{pmatrix} -2iC_- \tilde{\kappa} \mathbf{k} & -i(C_0^2 - C_0 D_- - 6\tilde{\kappa} \mathbf{k}) & -2C_- \tilde{\kappa} \mathbf{k} & -(C_0^2 - C_0 D_- - 6\tilde{\kappa} \mathbf{k}) \\ 2iC_- \tilde{\kappa} \mathbf{k} & i(C_0^2 + C_0 D_- - 6\tilde{\kappa} \mathbf{k}) & 2C_- \tilde{\kappa} \mathbf{k} & C_0^2 + C_0 D_- - 6\tilde{\kappa} \mathbf{k} \\ -2iC_- \tilde{\kappa} \mathbf{k} & i(C_0^2 - C_0 D_+ + 6\tilde{\kappa} \mathbf{k}) & 2C_- \tilde{\kappa} \mathbf{k} & -(C_0^2 - C_0 D_+ + 6\tilde{\kappa} \mathbf{k}) \\ 2iC_- \tilde{\kappa} \mathbf{k} & -i(C_0^2 + C_0 D_+ + 6\tilde{\kappa} \mathbf{k}) & -2C_- \tilde{\kappa} \mathbf{k} & C_0^2 + C_0 D_+ + 6\tilde{\kappa} \mathbf{k} \end{pmatrix}.$$

We now apply the “hydrodynamic” analysis [34, 35] with small frequencies for the equations of motion. Compared with the finite temperature case [28], in the extremal zero temperature case, the small ω expansion does not work at the horizon which gives an irregular singularity. A sensible way has been developed in [30] (see also [36]). In [37], Einstein-Maxwell theory on RN-AdS₄ background has been considered. In the low frequencies, we divide the holographic space labeled by u into two parts so-called “inner” and “outer” regions and solve the equations of motion independently. Then we match these solutions in the intermediate overlapping region to obtain full solutions.

B. Inner and outer regions, and matching

We first consider the equations of motion in the near horizon region. By using the coordinate redefinitions (8), the equations of motion (12) can be reduced to

$$0 = \tilde{\Theta}_a'' + \left(\omega_\tau^2 - \frac{1}{\zeta^2} \left(1 + 3\mathbf{k}^2 - \frac{1}{6} \mathcal{D}_{\tilde{\kappa}a}(\mathbf{k}) \right) \right) \tilde{\Theta}_a + \mathcal{O}(\epsilon), \quad (17)$$

where we have defined the frequency in AdS₂ spacetime as

$$\omega_\tau \equiv \frac{\omega}{\epsilon}, \quad (18)$$

due to the scaling of the time coordinate in (8). In order to keep ω_τ finite, the frequency ω should be small i.e. $\omega < \epsilon$. We now define the inner region so that the equations of motion (17) could be well approximated:

$$\text{Inner: } 1 - u < \epsilon \ll 1, \quad \text{with } \omega < \epsilon. \quad (19)$$

On the other hand, the outer region could be specified by introducing $\epsilon' (\gg \omega)$:

$$\text{Outer: } \omega \ll \epsilon' < 1 - u, \quad (20)$$

in which the first term in the potential (13) would be negligible compared to the other terms. Then the equations

of motion (12) can be reduced in this outer region:

$$0 = \tilde{\Theta}_a'' + \frac{\left(u^2(1-u)^2(1+2u) \right)'}{u^2(1-u)^2(1+2u)} \tilde{\Theta}_a' + \frac{(\mathcal{D}_{\tilde{\kappa}a}(\mathbf{k}) - 6)u - 18\mathbf{k}^2}{2u(1-u)^2(1+2u)} \tilde{\Theta}_a + \mathcal{O}(\omega). \quad (21)$$

In each regions, one can develop the solutions as a power series of ω .

The overlapping region could be given through the double scaling limit,

$$\omega \ll \epsilon' < 1 - u < \epsilon \ll 1. \quad (22)$$

One can match two solutions in the inner/outer in this overlapping region and may obtain full solutions.

We now proceed to solve the equations of motion in each regions and to perform their matching. In the inner region, the leading equations in (17) are equivalent to that for massive scalar fields in AdS₂ spacetime. The effective AdS₂ masses are

$$l^2 m_a^2 = 1 + 3\mathbf{k}^2 - \frac{1}{6} \mathcal{D}_{\tilde{\kappa}a}(\mathbf{k}). \quad (23)$$

This coincides with that discussed in [38]. In AdS spacetime, it is known that the mass is bounded below. This so-called Breitenlohner-Freedman (BF) bound provides the critical value of the CS coupling [38]. We will discuss this instability issue later.

Using the GKP-W relation [12], the conformal dimensions δ of operators which couple to the sources $\tilde{\Theta}_a(u)$ are given by

$$\delta_a = \frac{1}{2} + \Delta_{\tilde{\kappa}}^a(\mathbf{k}), \quad (24)$$

with

$$\begin{aligned} \Delta_{\tilde{\kappa}}^a(\mathbf{k}) &\equiv \sqrt{\left(\frac{1}{2} \right)^2 + l^2 m_a^2} \\ &= \frac{1}{2} \sqrt{5 + 12\mathbf{k}^2 - \frac{2}{3} \mathcal{D}_{\tilde{\kappa}a}(\mathbf{k})}. \end{aligned} \quad (25)$$

We can solve the leading equations of motion in (17) analytically

$$\tilde{\Theta}_{1a}^{(0)}(\zeta) = A_a \sqrt{\zeta} J_{\Delta_{\tilde{\kappa}}^a(\mathbf{k})}(\omega_\tau \zeta) + B_a \sqrt{\zeta} N_{\Delta_{\tilde{\kappa}}^a(\mathbf{k})}(\omega_\tau \zeta), \quad (26)$$

where $J_n(x)$ and $N_n(x)$ are Bessel functions of the first and second kinds, respectively. One of the integration constants A_a and B_a can be fixed by imposing the incoming wave condition at the horizon $\zeta \rightarrow \infty$ where is deep inner region. Then we obtain

$$\tilde{\Theta}_{1a}^{(0)}(\zeta) = C_a \sqrt{\zeta} \left(J_{\Delta_{\tilde{\kappa}}^a(\mathbf{k})}(\omega_\tau \zeta) + i N_{\Delta_{\tilde{\kappa}}^a(\mathbf{k})}(\omega_\tau \zeta) \right). \quad (27)$$

It should be noted that in the matching region (22) the condition $\omega_\tau \zeta \ll 1$ would be satisfied. Therefore in the matching region, we obtain the following asymptotic form of (27),

$$\tilde{\Theta}_{1a}^{(0)}(\zeta) \sim D_a \left\{ (1-u)^{-\frac{1}{2} + \Delta_{\tilde{\kappa}}^a(\mathbf{k})} + \mathcal{G}_{\mathbf{k}\tilde{\kappa}a}(\omega) (1-u)^{-\frac{1}{2} - \Delta_{\tilde{\kappa}}^a(\mathbf{k})} \right\}, \quad (28)$$

where we have used the relation given by (8) and (18)

$$\omega_\tau \zeta = \frac{\omega}{1-u}, \quad (29)$$

to get expressions in the original u -coordinate and also introduced the overall normalization constant D_a . Here $\mathcal{G}_{\mathbf{k}\tilde{\kappa}a}(\omega)$ can be estimated as

$$\begin{aligned} \mathcal{G}_{\mathbf{k}\tilde{\kappa}a}(\omega) &= -e^{-i\pi\Delta_{\tilde{\kappa}}^a(\mathbf{k})} \frac{\Gamma(1 - \Delta_{\tilde{\kappa}}^a(\mathbf{k}))}{\Gamma(1 + \Delta_{\tilde{\kappa}}^a(\mathbf{k}))} \left(\frac{\omega}{2} \right)^{2\Delta_{\tilde{\kappa}}^a(\mathbf{k})}, \end{aligned} \quad (30)$$

which may be related with the retarded correlation functions of the IR CFT [30],

$$G_{\text{IR}}(\omega, \mathbf{k})_{\tilde{\kappa}} \sim \Delta_{\tilde{\kappa}}^a(\mathbf{k}) \mathcal{G}_{\mathbf{k}\tilde{\kappa}a}(\omega). \quad (31)$$

It is easy to see that through the relation (29) the matching region (also inner region) covers the outer region. Hence we could expect to have an appropriate matching between the solutions.

In the outer region, we need to solve the leading equation of motion (21). From the right hand side of the inequality in (22), the near horizon region around $u = 1$ can be understood as the matching region from the outer region. We can factorize the regular singularity around the horizon as

$$\tilde{\Theta}_{0a}^{(0)}(u) = (1-u)^{\nu_a} F_a(u), \quad (32)$$

with

$$\nu_a = \nu_{a\pm} = -\frac{1}{2} \pm \Delta_{\tilde{\kappa}}^a(\mathbf{k}). \quad (33)$$

The function $F_a(u)$ follows the Heun equation [47],

$$\begin{aligned} 0 = F_a''(u) + \left\{ \frac{2}{u} + \frac{2(1+\nu_a)}{u-1} + \frac{1}{u+1/2} \right\} F_a'(u) \\ + \frac{\nu_a(\nu_a+4)u + \nu_a + \frac{9}{2}\mathbf{k}^2}{u(u-1)(u+1/2)} F_a(u). \end{aligned} \quad (34)$$

The Heun functions which are solutions of the Heun equation are not among the usual special functions. In order to solve the Heun equation in the entire region, one needs to use some numerical methods.

The exponents (33) are the same as those in (28). Therefore using the following functions

$$\eta_a^{(0)\pm}(u) = (1-u)^{-\frac{1}{2} \pm \Delta_{\tilde{\kappa}}^a(\mathbf{k})} \left(1 + \mathcal{O}(1-u) \right), \quad (35)$$

as two linear independent outer solutions in the matching region, the matching with the inner solution (28) could be carried out by fixing coefficients for linear combinations of the basis functions (35). Hence we have

$$\tilde{\Theta}_{0a}^{(0)}(u) = D_a \left\{ \eta_a^{(0)+}(u) + \mathcal{G}_{\mathbf{k}\tilde{\kappa}a}(\omega) \eta_a^{(0)-}(u) \right\}. \quad (36)$$

Near the boundary $u = 0$, the indicial exponents of (21) (and (34)) are $(0, -1)$. We take the following Frobenius series solutions as two independent basis,

$$\xi^{(\text{I})}(u) = 1 + \mathcal{O}(u), \quad (37)$$

$$\xi^{(\text{II})}(u) = \frac{1}{u} \left(1 + \mathcal{O}(u^2) \right) + 9\mathbf{k}^2 \xi^{(\text{I})}(u) \log u,$$

which correspond to the normalizable and nonnormalizable modes, respectively [39]. We can develop the solutions (36) to the boundary and may express those by the asymptotic expansion of the two basis solutions (37),

$$\begin{aligned} \tilde{\Theta}_{0a}^{(0)}(u) &= D_a \left\{ \left(\mathcal{A}_{\tilde{\kappa}a}^+(\mathbf{k}) \xi^{(\text{I})}(u) + \mathcal{B}_{\tilde{\kappa}a}^+(\mathbf{k}) \xi^{(\text{II})}(u) \right) \right. \\ &\quad \left. + \mathcal{G}_{\mathbf{k}\tilde{\kappa}a}(\omega) \left(\mathcal{A}_{\tilde{\kappa}a}^-(\mathbf{k}) \xi^{(\text{I})}(u) + \mathcal{B}_{\tilde{\kappa}a}^-(\mathbf{k}) \xi^{(\text{II})}(u) \right) \right\}. \end{aligned} \quad (38)$$

Although there are no analytic expressions of the connection coefficients $\mathcal{A}_{\tilde{\kappa}a}^\pm(\mathbf{k})$ and $\mathcal{B}_{\tilde{\kappa}a}^\pm(\mathbf{k})$, one could discuss the low frequency behavior of the correlation functions [30]. Following this direction, we proceed to obtain the correlation functions “formally”. One can also consider higher order corrections in the small ω -expansions [30]. The connection coefficients could be determined pertur-

batively,

$$\begin{aligned}
& \mathcal{A}_{\tilde{\kappa}a}^{\pm}(\mathbf{k}) \\
& \longrightarrow \mathcal{A}_{\tilde{\kappa}a}^{\pm}(\omega, \mathbf{k}) = \mathcal{A}_{\tilde{\kappa}a}^{\pm(0)}(\mathbf{k}) + \omega \mathcal{A}_{\tilde{\kappa}a}^{\pm(1)}(\mathbf{k}) + \mathcal{O}(\omega^2), \\
& \mathcal{B}_{\tilde{\kappa}a}^{\pm}(\mathbf{k}) \\
& \longrightarrow \mathcal{B}_{\tilde{\kappa}a}^{\pm}(\omega, \mathbf{k}) = \mathcal{B}_{\tilde{\kappa}a}^{\pm(0)}(\mathbf{k}) + \omega \mathcal{B}_{\tilde{\kappa}a}^{\pm(1)}(\mathbf{k}) + \mathcal{O}(\omega^2).
\end{aligned} \tag{39}$$

However, in this paper, it is enough to consider only the leading contributions.

The remaining thing is to fix the overall constants D_a

in the solutions. These can be determined in terms of the field values at the boundary. In order for that, we use relations derived by (10a) and (16),

$$\begin{aligned}
& u^2 \Theta'_{x(y)\pm} - u C_{\pm} B'_{x(y)} \Big|_{u=0} \\
& = 9 \left(\mathbf{k}^2 (h_t^{x(y)})^{(0)} + \omega \mathbf{k} (h_z^{x(y)})^{(0)} \right) - C_{\pm} (B_{x(y)})^{(0)},
\end{aligned}$$

where $(h_t^{x(y)})^{(0)}$, $(h_z^{x(y)})^{(0)}$ and $(B_{x(y)})^{(0)}$ stand for their constant values at the boundary. Hence we could obtain the overall constants as:

$$\begin{aligned}
D_1 &= \frac{9\tilde{K}_- \left(\mathbf{k}^2 (i(h_t^x)^{(0)} + (h_t^y)^{(0)}) + \omega \mathbf{k} (i(h_z^x)^{(0)} + (h_z^y)^{(0)}) \right) - C_- \tilde{K}_+ \left(i(B_x)^{(0)} + (B_y)^{(0)} \right)}{\left(\mathcal{B}_{\tilde{\kappa}1}^+(\mathbf{k}) + \mathcal{G}_{\mathbf{k}\tilde{\kappa}1}(\omega) \mathcal{B}_{\tilde{\kappa}1}^-(\mathbf{k}) \right)}, \\
D_2 &= \frac{-9K_- \left(\mathbf{k}^2 (i(h_t^x)^{(0)} + (h_t^y)^{(0)}) + \omega \mathbf{k} (i(h_z^x)^{(0)} + (h_z^y)^{(0)}) \right) + C_- K_+ \left(i(B_x)^{(0)} + (B_y)^{(0)} \right)}{\left(\mathcal{B}_{\tilde{\kappa}2}^+(\mathbf{k}) + \mathcal{G}_{\mathbf{k}\tilde{\kappa}2}(\omega) \mathcal{B}_{\tilde{\kappa}2}^-(\mathbf{k}) \right)}, \\
D_3 &= \frac{-9\tilde{L}_- \left(\mathbf{k}^2 (i(h_t^x)^{(0)} - (h_t^y)^{(0)}) + \omega \mathbf{k} (i(h_z^x)^{(0)} - (h_z^y)^{(0)}) \right) + C_- \tilde{L}_+ \left(i(B_x)^{(0)} - (B_y)^{(0)} \right)}{\left(\mathcal{B}_{\tilde{\kappa}3}^+(\mathbf{k}) + \mathcal{G}_{\mathbf{k}\tilde{\kappa}3}(\omega) \mathcal{B}_{\tilde{\kappa}3}^-(\mathbf{k}) \right)}, \\
D_4 &= \frac{9L_- \left(\mathbf{k}^2 (i(h_t^x)^{(0)} - (h_t^y)^{(0)}) + \omega \mathbf{k} (i(h_z^x)^{(0)} - (h_z^y)^{(0)}) \right) - C_- L_+ \left(i(B_x)^{(0)} - (B_y)^{(0)} \right)}{\left(\mathcal{B}_{\tilde{\kappa}4}^+(\mathbf{k}) + \mathcal{G}_{\mathbf{k}\tilde{\kappa}4}(\omega) \mathcal{B}_{\tilde{\kappa}4}^-(\mathbf{k}) \right)},
\end{aligned} \tag{40}$$

with

$$\begin{aligned}
K_{\pm}(\mathbf{k}, \tilde{\kappa}) &\equiv C_0^2(\mathbf{k}) + D_-(\tilde{\kappa}, \mathbf{k}) C_0(\mathbf{k}) \\
&\quad + 2\tilde{\kappa} \left(C_{\pm}(\mathbf{k}) - 3 \right) \mathbf{k}, \\
\tilde{K}_{\pm}(\mathbf{k}, \tilde{\kappa}) &\equiv C_0^2(\mathbf{k}) - D_-(\tilde{\kappa}, \mathbf{k}) C_0(\mathbf{k}) \\
&\quad + 2\tilde{\kappa} \left(C_{\pm}(\mathbf{k}) - 3 \right) \mathbf{k}, \\
L_{\pm}(\mathbf{k}, \tilde{\kappa}) &\equiv C_0^2(\mathbf{k}) + D_+(\tilde{\kappa}, \mathbf{k}) C_0(\mathbf{k}) \\
&\quad - 2\tilde{\kappa} \left(C_{\pm}(\mathbf{k}) - 3 \right) \mathbf{k}, \\
\tilde{L}_{\pm}(\mathbf{k}, \tilde{\kappa}) &\equiv C_0^2(\mathbf{k}) - D_+(\tilde{\kappa}, \mathbf{k}) C_0(\mathbf{k}) \\
&\quad - 2\tilde{\kappa} \left(C_{\pm}(\mathbf{k}) - 3 \right) \mathbf{k}.
\end{aligned} \tag{41}$$

We have formally fixed the leading order solutions of the master variables $\tilde{\Theta}_a^{(0)}(u)$.

IV. RETARDED CORRELATION FUNCTIONS

In order to obtain the two-point retarded correlation function via the GKP-W relation [12], we need the

bilinear-part of the regularized on-shell action at the boundary $u = 0$. Here we can prepare the following counter terms for the UV regularization [40]:

$$S_{\text{ct}} = S_{\text{ct}}^{\text{gravity}} + S_{\text{ct}}^{\text{gauge}}, \tag{42}$$

with

$$\begin{aligned}
S_{\text{ct}}^{\text{gravity}} &= \frac{1}{8\pi G_5} \left\{ \int d^4x \sqrt{-g^{(4)}} \left(\frac{3}{l} + \frac{l}{4} R^{(4)} \right) \right. \\
&\quad \left. - \frac{l^3}{16} \log u \int d^4x \sqrt{-g^{(4)}} \left(R_{\mu\nu}^{(4)} R^{(4)\mu\nu} - \frac{1}{3} (R^{(4)})^2 \right) \right\},
\end{aligned}$$

$$S_{\text{ct}}^{\text{gauge}} = \frac{l}{8e^2} \log u \int d^4x \sqrt{-g^{(4)}} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu},$$

where $R^{(4)}$ and $R_{\mu\nu}^{(4)}$ are the scalar curvature and the Ricci tensor on the 4D boundary, respectively.

The bilinear parts of the perturbations in the bulk actions (2a), (2c) and (2d) are reduced to the surface

terms by using the equations of motion. In the present gauge, the extrinsic curvature in the Gibbons-Hawking term (2b) is given as $K = g^{(4)\mu\nu} \partial_u g_{\mu\nu}^{(4)} / (2\sqrt{g_{uu}})$. Then

in the low frequencies, the regularized boundary action is given by

$$\begin{aligned}
S_{\text{on-shell}} &= \lim_{u \rightarrow 0} (S + S_{\text{ct}}) \Big|_{\text{on-shell}} \\
&= \lim_{u \rightarrow 0} \frac{l^3}{256\pi b^4 G_5} \int \frac{d^2 k}{(2\pi)^2} \left\{ \frac{1}{u} \left(h_t^x(-k, u) h_t^{x'}(k, u) - h_z^x(-k, u) h_z^{x'}(k, u) \right) \right. \\
&\quad + \frac{3}{2} \left(3h_t^x(-k, u) h_t^x(k, u) + h_z^x(-k, u) h_z^x(k, u) \right) \\
&\quad - 6B_x(-k, u) B_x'(k, u) + 6B_x(-k, u) h_t^x(k, u) \\
&\quad + \left(\frac{9}{u} + 81\mathbf{k}^2 \log u \right) \\
&\quad \times \left(\mathbf{k}^2 h_t^x(-k, u) h_t^x(k, u) + \omega^2 h_z^x(-k, u) h_z^x(k, u) + 2\omega \mathbf{k} h_t^x(-k, u) h_z^x(k, u) \right) \\
&\quad \left. + 54\mathbf{k}^2 \log u B_x(-k, u) B_x(k, u) + (x \rightarrow y) \right\}. \tag{44}
\end{aligned}$$

Performing the matrix inversion (15) and (16) to rewrite the solutions in terms of the original variables, and following the prescription given in Appendix, we could obtain two-point retarded correlation functions:

$$G_{xx}(\omega, \mathbf{k})_{\tilde{\kappa}} = -\frac{l}{12e^2 b^2} \left\{ 54\mathbf{k}^2 + \frac{3}{4\tilde{\kappa} C_0^2} \tilde{G}_{1+}(\omega, \mathbf{k})_{\tilde{\kappa}} \right\}, \tag{45a}$$

$$G_{xy}(\omega, \mathbf{k})_{\tilde{\kappa}} = \frac{l}{12e^2 b^2} \left(\frac{3i}{8\tilde{\kappa} C_0^2} \right) \tilde{G}_{2+}(\omega, \mathbf{k})_{\tilde{\kappa}}, \tag{45b}$$

$$\begin{aligned}
G_{xt}(\omega, \mathbf{k})_{\tilde{\kappa}} &= \frac{l^3}{128\pi b^4 G_5} \left\{ \frac{9}{2} - 81\mathbf{k}^4 \right. \\
&\quad \left. + \frac{9\mathbf{k}}{8\tilde{\kappa} C_0^2} \tilde{G}_{1-}(\omega, \mathbf{k})_{\tilde{\kappa}} \right\}, \tag{45c}
\end{aligned}$$

$$\begin{aligned}
G_{xz}(\omega, \mathbf{k})_{\tilde{\kappa}} &= \frac{l^3}{128\pi b^4 G_5} \left\{ \frac{3}{2} - 81\omega^2 \mathbf{k}^2 \right. \\
&\quad \left. + \frac{9\omega^2}{8\tilde{\kappa} C_0^2} \tilde{G}_{1-}(\omega, \mathbf{k})_{\tilde{\kappa}} \right\}, \tag{45d}
\end{aligned}$$

$$\begin{aligned}
G_{xt}(\omega, \mathbf{k})_{\tilde{\kappa}} &= \frac{l^3}{256\pi b^4 G_5} \\
&\quad \times \left\{ 81\omega \mathbf{k}^4 - \frac{9\omega}{8\tilde{\kappa} C_0^2} \tilde{G}_{1-}(\omega, \mathbf{k})_{\tilde{\kappa}} \right\}, \tag{45e}
\end{aligned}$$

$$G_{xt}(\omega, \mathbf{k})_{\tilde{\kappa}} = -\frac{l^3}{256\pi b^4 G_5} \left(\frac{9i\mathbf{k}}{8\tilde{\kappa} C_0^2} \right) \tilde{G}_{2-}(\omega, \mathbf{k})_{\tilde{\kappa}}, \tag{45f}$$

$$G_{xt}(\omega, \mathbf{k})_{\tilde{\kappa}} = \frac{l^3}{256\pi b^4 G_5} \left(\frac{9i\omega}{8\tilde{\kappa} C_0^2} \right) \tilde{G}_{2-}(\omega, \mathbf{k})_{\tilde{\kappa}}, \tag{45g}$$

$$G_{xz}(\omega, \mathbf{k})_{\tilde{\kappa}} = -\frac{l^3}{256\pi b^4 G_5} \left(\frac{9i\omega^2}{8\tilde{\kappa} C_0^2} \right) \tilde{G}_{2-}(\omega, \mathbf{k})_{\tilde{\kappa}}, \tag{45h}$$

$$\begin{aligned}
G_{xt}(\omega, \mathbf{k})_{\tilde{\kappa}} &= -\frac{l^2}{32e\sqrt{6\pi} G_5 b^3} \\
&\quad \times \left\{ 6 + \frac{27\mathbf{k}}{4\tilde{\kappa} C_0^2 C_-} \tilde{G}_{3-}(\omega, \mathbf{k})_{\tilde{\kappa}} \right\}, \tag{45i}
\end{aligned}$$

$$G_{xz}(\omega, \mathbf{k})_{\tilde{\kappa}} = \frac{l^2}{32e\sqrt{6\pi} G_5 b^3} \left(\frac{27\omega}{4\tilde{\kappa} C_0^2 C_-} \right) \tilde{G}_{3-}(\omega, \mathbf{k})_{\tilde{\kappa}}, \tag{45j}$$

$$G_{yt}(\omega, \mathbf{k})_{\tilde{\kappa}} = \frac{l^3}{32e\sqrt{6\pi} G_5 b^3} \left(\frac{27i\mathbf{k}}{4\tilde{\kappa} C_0^2 C_-} \right) \tilde{G}_{3+}(\omega, \mathbf{k})_{\tilde{\kappa}}, \tag{45k}$$

$$G_{yz}(\omega, \mathbf{k})_{\tilde{\kappa}} = -\frac{l^2}{32e\sqrt{6\pi} G_5 b^3} \left(\frac{27i\omega}{4\tilde{\kappa} C_0^2 C_-} \right) \tilde{G}_{3+}(\omega, \mathbf{k})_{\tilde{\kappa}}, \tag{45l}$$

with

$$\begin{aligned}
\tilde{G}_{1\pm}(\omega, \mathbf{k})_{\tilde{\kappa}} &= \left(\frac{\tilde{K}_{\pm} K_{\pm}}{D_{-}} \right) G_1(\omega, \mathbf{k})_{\tilde{\kappa}} - \left(\frac{\tilde{K}_{\mp} K_{\pm}}{D_{-}} \right) G_2(\omega, \mathbf{k})_{\tilde{\kappa}} \\
&\quad - \left(\frac{\tilde{L}_{\pm} L_{\pm}}{D_{+}} \right) G_3(\omega, \mathbf{k})_{\tilde{\kappa}} + \left(\frac{\tilde{L}_{\mp} L_{\pm}}{D_{+}} \right) G_4(\omega, \mathbf{k})_{\tilde{\kappa}}, \\
\tilde{G}_{2\pm}(\omega, \mathbf{k})_{\tilde{\kappa}} &= \left(\frac{\tilde{K}_{\pm} K_{\pm}}{D_{-}} \right) G_1(\omega, \mathbf{k})_{\tilde{\kappa}} - \left(\frac{\tilde{K}_{\mp} K_{\pm}}{D_{-}} \right) G_2(\omega, \mathbf{k})_{\tilde{\kappa}} \\
&\quad + \left(\frac{\tilde{L}_{\pm} L_{\pm}}{D_{+}} \right) G_3(\omega, \mathbf{k})_{\tilde{\kappa}} - \left(\frac{\tilde{L}_{\mp} L_{\pm}}{D_{+}} \right) G_4(\omega, \mathbf{k})_{\tilde{\kappa}}, \\
\tilde{G}_{3\pm}(\omega, \mathbf{k})_{\tilde{\kappa}} &= \left(\frac{\tilde{K}_{-} K_{-}}{D_{-}} \right) (G_1(\omega, \mathbf{k})_{\tilde{\kappa}} - G_2(\omega, \mathbf{k})_{\tilde{\kappa}}) \\
&\quad \pm \left(\frac{\tilde{L}_{-} L_{-}}{D_{+}} \right) (G_3(\omega, \mathbf{k})_{\tilde{\kappa}} - G_4(\omega, \mathbf{k})_{\tilde{\kappa}}),
\end{aligned}$$

and

$$\begin{aligned}
G_a(\omega, \mathbf{k})_{\tilde{\kappa}} &= \frac{\mathcal{A}_{\tilde{\kappa}a}^+(\mathbf{k}) + \mathcal{G}_{\mathbf{k}\tilde{\kappa}a}(\omega)\mathcal{A}_{\tilde{\kappa}a}^-(\mathbf{k})}{\mathcal{B}_{\tilde{\kappa}a}^+(\mathbf{k}) + \mathcal{G}_{\mathbf{k}\tilde{\kappa}a}(\omega)\mathcal{B}_{\tilde{\kappa}a}^-(\mathbf{k})} \\
&\sim \frac{\mathcal{A}_{\tilde{\kappa}a}^+(\mathbf{k})}{\mathcal{B}_{\tilde{\kappa}a}^+(\mathbf{k})} \left\{ 1 + \left(\frac{\mathcal{A}_{\tilde{\kappa}a}^-(\mathbf{k})}{\mathcal{A}_{\tilde{\kappa}a}^+(\mathbf{k})} - \frac{\mathcal{B}_{\tilde{\kappa}a}^-(\mathbf{k})}{\mathcal{B}_{\tilde{\kappa}a}^+(\mathbf{k})} \right) \mathcal{G}_{\mathbf{k}\tilde{\kappa}a}(\omega) + \dots \right\},
\end{aligned} \tag{46}$$

where K_{\pm} , \tilde{K}_{\pm} , L_{\pm} and \tilde{L}_{\pm} are given in (41). In the last line in (46), we did small ω expansions assuming $\mathcal{B}_{\tilde{\kappa}a}^+ \neq 0$. We can observe non-analytical frequency behavior of the correlation functions which is determined by the IR CFT i.e.

$$\mathcal{G}_{\mathbf{k}\tilde{\kappa}a}(\omega) \propto \omega^{2\Delta_{\tilde{\kappa}}^a(\mathbf{k})}. \tag{47}$$

UV informations are given by the coefficients $\mathcal{A}_{\tilde{\kappa}a}^{\pm}(\mathbf{k})$ and $\mathcal{B}_{\tilde{\kappa}a}^{\pm}(\mathbf{k})$; more generally $\mathcal{A}_{\tilde{\kappa}a}^{\pm}(\omega, \mathbf{k})$ and $\mathcal{B}_{\tilde{\kappa}a}^{\pm}(\omega, \mathbf{k})$ in (39) [30].

V. DISCUSSION

It has been discussed that the presence of the CS term can cause the uniform solution without the CS term to be unstable against non-uniform current ordering above the critical value $\tilde{\kappa}_c$ determined from the Breitenlohner-Freedman (BF) bound for fluctuations [38]. Indeed, the CS term gives rise to couplings between the x - and y - components of the “effective” vector potential with “anomalous” correlations. This indicates that dual $U(1)$ charge currents may feel anomalous interactions between not only x - or y - components but also x - and y -components, allowing us to expect possible instability against non-uniform current ordering. If the CS term is turned off, such correlations between effective gauge fields disappear and their correlations become trivial. Although the CS coefficient cannot be used as a tuning parameter, we speculate that the corresponding CFT with $\tilde{\kappa} \approx \tilde{\kappa}_c$ differs from that with $\tilde{\kappa} < \tilde{\kappa}_c$ in the respect that the CFT of $\tilde{\kappa} < \tilde{\kappa}_c$ may describe uniform current fluctuations from the trivial vacuum state while that of $\tilde{\kappa} \approx \tilde{\kappa}_c$ would describe complex current fluctuations with anomalous correlations. It is interesting to observe that the CS coefficient given by the string theory is almost identical to the critical value [38].

An essential question is the role of the CS term in critical exponents of $\mathcal{G}_{\mathbf{k}\tilde{\kappa}a}(\omega)$, i.e. $\Delta_{\tilde{\kappa}}^a(\mathbf{k})$ given by (25). In the absence of the CS term we obtain $\Delta_{\tilde{\kappa}=0}^1(\mathbf{k}) = \Delta_{\tilde{\kappa}=0}^3(\mathbf{k}) > \Delta_{\tilde{\kappa}=0}^2(\mathbf{k}) = \Delta_{\tilde{\kappa}=0}^4(\mathbf{k})$ as shown in the right panel of Fig.1. This is quite natural because the conformal dimensions of the x -current is expected to be the same as that of the y -current. On the other hand, the CS term makes $\Delta_{\tilde{\kappa}=0}^1(\mathbf{k}) = \Delta_{\tilde{\kappa}=0}^3(\mathbf{k})$ split into $\Delta_{\tilde{\kappa}}^1(\mathbf{k}) < \Delta_{\tilde{\kappa}=0}^1(\mathbf{k}) = \Delta_{\tilde{\kappa}=0}^3(\mathbf{k}) < \Delta_{\tilde{\kappa}}^3(\mathbf{k})$. The same thing happens for $\Delta_{\tilde{\kappa}=0}^2(\mathbf{k}) = \Delta_{\tilde{\kappa}=0}^4(\mathbf{k})$. This becomes

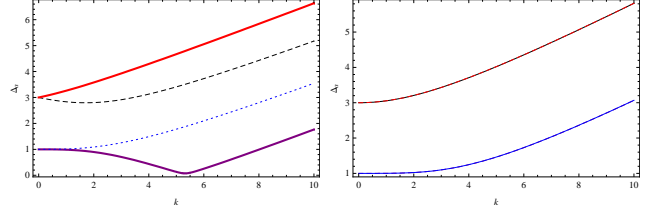


FIG. 1: Left : Critical exponents in the presence of the CS term, showing $\Delta_{\tilde{\kappa}}^2(\mathbf{k}) < \Delta_{\tilde{\kappa}}^4(\mathbf{k}) < \Delta_{\tilde{\kappa}}^1(\mathbf{k}) < \Delta_{\tilde{\kappa}}^3(\mathbf{k})$. The minimum value of $\Delta_{\tilde{\kappa}}^2(\mathbf{k}_c)$ is associated with the instability to the non-uniform current ordering. Right : Critical exponents in the absence of the CS term, showing $\Delta_{\tilde{\kappa}=0}^2(\mathbf{k}) = \Delta_{\tilde{\kappa}=0}^4(\mathbf{k}) < \Delta_{\tilde{\kappa}=0}^1(\mathbf{k}) = \Delta_{\tilde{\kappa}=0}^3(\mathbf{k})$.

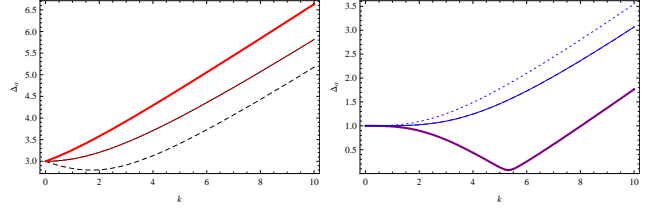


FIG. 2: Left : $\Delta_{\tilde{\kappa}=0}^1(\mathbf{k}) = \Delta_{\tilde{\kappa}=0}^3(\mathbf{k})$ are split to $\Delta_{\tilde{\kappa}}^1(\mathbf{k}) < \Delta_{\tilde{\kappa}=0}^1(\mathbf{k}) = \Delta_{\tilde{\kappa}=0}^3(\mathbf{k}) < \Delta_{\tilde{\kappa}}^3(\mathbf{k})$. Right : $\Delta_{\tilde{\kappa}=0}^2(\mathbf{k}) = \Delta_{\tilde{\kappa}=0}^4(\mathbf{k})$ are separated into $\Delta_{\tilde{\kappa}}^2(\mathbf{k}) < \Delta_{\tilde{\kappa}=0}^2(\mathbf{k}) = \Delta_{\tilde{\kappa}=0}^4(\mathbf{k}) < \Delta_{\tilde{\kappa}}^4(\mathbf{k})$.

$\Delta_{\tilde{\kappa}}^2(\mathbf{k}) < \Delta_{\tilde{\kappa}=0}^2(\mathbf{k}) = \Delta_{\tilde{\kappa}=0}^4(\mathbf{k}) < \Delta_{\tilde{\kappa}}^4(\mathbf{k})$ by the CS term. These are shown in the left panel of Fig.2. As discussed before, the presence of the CS term is expected to cause the instability to the non-uniform current ordering. This effect can be seen from the fact that the BF bound is translated into the minimum value of $\Delta_{\tilde{\kappa}}^2(\mathbf{k}_c)$, as shown in the left panel of Fig.1. As a result, we find $\Delta_{\tilde{\kappa}}^2(\mathbf{k}_c) < \Delta_{\tilde{\kappa}=0}^2(\mathbf{k}_c) = \Delta_{\tilde{\kappa}=0}^4(\mathbf{k}_c) < \Delta_{\tilde{\kappa}}^4(\mathbf{k}_c) < \Delta_{\tilde{\kappa}}^1(\mathbf{k}_c) < \Delta_{\tilde{\kappa}=0}^1(\mathbf{k}_c) = \Delta_{\tilde{\kappa}=0}^3(\mathbf{k}_c) < \Delta_{\tilde{\kappa}}^3(\mathbf{k}_c)$ for critical fluctuations. Novel critical exponents appear in the current-current correlation function of critical fluctuations as a result of the interplay between the CS term and the emergent locality.

We interpret the emergence of novel critical exponents as $U(1)$ charge fractionalization. The $U(1)$ current with the conformal dimension $\Delta_{\tilde{\kappa}=0}^1(\mathbf{k}_c)$ becomes fractionalized into that with $\Delta_{\tilde{\kappa}}^1(\mathbf{k}_c)$, where the latter $U(1)$ current carries smaller $U(1)$ charge and has smaller conformal dimension. Then, the other $U(1)$ current with $\Delta_{\tilde{\kappa}}^3(\mathbf{k}_c)$ may be identified with some composites of the $U(1)$ current with $\Delta_{\tilde{\kappa}}^1(\mathbf{k}_c)$, where $\Delta_{\tilde{\kappa}}^3(\mathbf{k}_c)$ should be larger than $\Delta_{\tilde{\kappa}}^1(\mathbf{k}_c)$, satisfied indeed. Essentially the same physics is also applied to $\Delta_{\tilde{\kappa}}^2(\mathbf{k}_c)$ and $\Delta_{\tilde{\kappa}}^4(\mathbf{k}_c)$.

It is important to understand the role of the AdS_2 geometry at low energies. If we do not take the extremal limit of the RN-AdS_5 , the CS term does not modify any anomalous critical exponents, where its role turns out to result in additional analytic expansions for momentum and frequency [28]. However, the CS term in the emergent AdS_2 geometry generates novel critical exponents, which will be associated with novel excitations.

The AdS_2 geometry reminds us of the dynamical mean-field theory (DMFT) framework [41] as the dual CFT, where the extended DMFT of the Kondo lattice model gives rise to the local critical theory [42]. Recently, Sachdev proposed the critical field theory of the disordered Kondo-Heisenberg model as the CFT dual to the AdS_2 gravity theory with fermionic or bosonic matters [27]. One of the authors has suggested that symmetry associated with charge and spin fluctuations becomes enlarged at the local quantum critical point and this emergent enhanced symmetry allows us to assign a nontrivial quantum number to an instanton excitation [43]. These topological excitations are identified with spinons and holons, respectively. Unfortunately, it is not clear at all how the symmetry enhancement appears to allow topological excitations in the AdS_2 gravity theory [27].

The present dual gravity theory does not have either fermionic or bosonic matters. As discussed in the introduction, the corresponding conformal field theory will be given in terms of strongly coupled $U(1)$ charge currents with the associated chiral anomaly. In this respect the origin of the locality is not clearly figured out. Although the circulating current model [44] attracts our interest, the connection with the AdS_2 dual gravity is not clear.

The interplay between interaction and θ vacua has been discussed, based on the EM-CS theory in AdS_3 [45]. The chiral anomaly for symmetry currents in (1+1)D CFTs turns out to determine their correlators completely, where a perfect metallic state appears to be distinguished from the present RN- AdS_5 case, but consistent with the chiral edge-state picture. On the other hand, the absence of the CS term, i.e. the Einstein-Maxwell (EM) theory is interpreted to be the presence of an external current coupled with the EM gauge field, where the external current serves an additional source for the Weyl anomaly.

It will be interesting to investigate the Yang-Mills-Chern-Simons theory on the RN- AdS_5 in the extremal limit, where the corresponding conformal field theory is expected to be in terms of strongly correlated non-abelian currents with the chiral anomaly. Recently, the nonabelian gauge theory with the θ term has been suggested for fractional magnetoelectric effect in interacting topological insulators [26], where dyon excitations carrying fractional electric charge are responsible for such an effect. Although we cannot estimate the role of supersymmetry and emergent locality, we speculate that some types of quantum criticality in such topological insulators may be associated with a generalized framework of the present description.

VI. SUMMARY

The interplay between correlations and topological terms has proposed novel quantum states of matter and emergent excitations beyond our intuition. However, it was outside the field theoretical framework to incorporate both nonperturbative quantum effects reliably above one

dimension. This motivated us to investigate the Einstein-Maxwell-Chern-Simons theory on Reissner-Nordström- AdS_5 background, where the corresponding field theory is expected to be in terms of strongly interacting $U(1)$ charge currents with anomalous “chiral” currents.

An important aspect was the interplay between the Chern-Simons term and the emergent AdS_2 geometry of the extremal limit. If we do not take the extremal limit, the Chern-Simons term itself does not generate any anomalous critical exponents in current-current correlation functions, where only analytic expansions appear for momentum and frequency. If we do not introduce the Chern-Simons term in the AdS_2 geometry, critical exponents become rather trivial, where the non-analyticity results from the emergent locality.

We interpret the emergence of novel critical exponents as a result of complicated current-pattern fluctuations, driven by the Chern-Simons term. The Chern-Simons term gives rise to couplings between x - and y -components of gauge fields with anomalous correlations. Such anomalous correlations are expected to cause nontrivial current-current interactions in different directions, which may be responsible for non-uniform current-loop excitations. On the other hand, the absence of the Chern-Simons term does not result in complicated current patterns, where both x - and y -directional current fluctuations have the same conformal dimensions.

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Appendix: Minkowskian correlators in AdS/CFT correspondence

We here briefly summarize the prescription for the Minkowskian correlator in AdS/CFT correspondence. We here follow the prescription proposed in [34]. We work on the following 5D background,

$$(ds)^2 = g_{\mu\nu}dx^\mu dx^\nu + g_{uu}(du)^2, \quad (\text{A.1})$$

where x^μ and u are the 4D and radial coordinates, respectively. We refer the boundary at $u = 0$ and the horizon at $u = 1$. Let us consider a solution of an equation of

motion in this 5D background. Suppose a solution of an equation of motion is given by

$$\phi(u, x) = \int \frac{d^4 k}{(2\pi)^4} e^{ikx} f_k(u) \phi^0(k), \quad (\text{A.2})$$

where $f_k(u)$ is normalized such that $f_k(0) = 1$ at the boundary. After putting the equation of motion back into the action, the on shell action might be reduced to surface terms

$$S[\phi^0] = \int \frac{d^4 k}{(2\pi)^4} \phi^0(-k) \mathcal{G}(k, u) \phi^0(k) \Big|_{u=1}^{u=0}. \quad (\text{A.3})$$

Here, the function $\mathcal{G}(k, u)$ can be written in terms of $f_{\pm k}(u)$ and $\partial_u f_{\pm k}(u)$. Accommodating GKP-W relation [12] to Minkowski spacetime, Son and Starinets proposed the formula to get the retarded correlation functions,

$$G(k) = 2\mathcal{G}(k, u) \Big|_{u=0}. \quad (\text{A.4})$$

We define the retarded correlation function we discuss in this paper:

$$\begin{aligned} G_{\mu\nu}(\omega, k) &= -i \int \frac{d^2 x}{(2\pi)^2} e^{-i\omega t + ikz} \theta(t) \langle [J_\mu(t, z), J_\nu(0, 0)] \rangle, \\ G_{\mu\nu\rho\sigma}(\omega, k) &= -i \int \frac{d^2 x}{(2\pi)^2} e^{-i\omega t + ikz} \theta(t) \langle [T_{\mu\nu}(t, z), T_{\rho\sigma}(0, 0)] \rangle, \\ G_{\mu\rho\sigma}(\omega, k) &= -i \int \frac{d^2 x}{(2\pi)^2} e^{-i\omega t + ikz} \theta(t) \langle [J_\mu(t, z), T_{\rho\sigma}(0, 0)] \rangle, \end{aligned} \quad (\text{A.5})$$

where the momentum in this paper is taken to z -direction, and the operators $J_\mu(t, z)$ and $T_{\mu\nu}(t, z)$ are the $U(1)$ current and the energy-momentum tensor, respectively.

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- $$0 = \mathcal{W}'' + \left\{ \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a} \right\} \mathcal{W}' + \frac{\alpha\beta z - q}{z(z-1)(z-a)} \mathcal{W},$$
- with $\epsilon = \alpha + \beta - \gamma - \delta + 1$. Under the assignment of the parameters;
- $$\alpha = \nu_a, \quad \beta = \nu_a + 4, \quad \gamma = 2, \quad \delta = 2(1 + \nu_a),$$
- $$a = -\frac{1}{2}, \quad q = -\nu_a - \frac{9}{2}k^2,$$
- we could have the form (34).